All of the properties of integer exponents also apply to rational exponents. Here is a summary.

Summary — **Properties of Rational Exponents**

Let m and n represent rational numbers. Assume that no denominator equals 0.

Property	Example
$a^m \cdot a^n = a^{m+n}$	$8^{\frac{1}{3}} \cdot 8^{\frac{2}{3}} = 8^{\frac{1}{3}+\frac{2}{3}} = 8^1 = 8$
$(a^m)^n = a^{mn}$	$\left(5^{\frac{1}{2}}\right)^4 = 5^{\frac{1}{2} \cdot 4} = 5^2 = 25$
$(ab)^m = a^m b^m$	$(4 \cdot 5)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 2 \cdot 5^{\frac{1}{2}}$
$a^{-m} = \frac{1}{a^m}$	$9^{-\frac{1}{2}} = \frac{1}{9^{\frac{1}{2}}} = \frac{1}{3}$
$\frac{a^m}{a^n} = a^{m-n}$	$\frac{\pi^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} = \pi^{\frac{3}{2}-\frac{1}{2}} = \pi^1 = \pi$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{5}{27}\right)^{\frac{1}{3}} = \frac{5^{\frac{1}{3}}}{27^{\frac{1}{3}}} = \frac{5^{\frac{1}{3}}}{3}$

You can simplify a number with a rational exponent by using the properties of exponents or by converting the expression to a radical expression.

4 EXAMPLE Simplifying Numbers With Rational Exponents

Simplify each number.

a. $(-32)^{\frac{3}{5}}$

Method 1

Method 2

Introduction to Functions

Learning Objectives:

1. Understand the definition of a relation.
2. Understand the definition of a function.
3. Decide whether an equation defines a function.
4. Use function notation.
5. Apply function concepts in applications.

Examples:

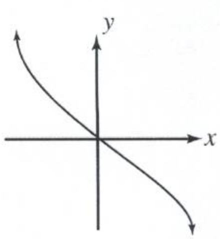
1. Complete the table for the function defined by $f(x) = x - 3$.

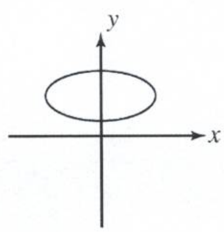
	x	$x-3$	$f(x)$	(x, y)
a	-3			
b	-2			
c	-1			
d	0			
e	1			

2. Determine the domain, the range, and whether each relation represents a function.

a) $\{(1,3),(2,5),(4,1)\}$ b) $\{(-1,3),(1,3),(2,-5)\}$ c) $\{(7,-1),(3,-1),(7,4)\}$

3. Decide whether each relation represents a function.

a) 

b) 

c) $y = 7x - 3$

d) $x = y^2 + 5$

4. For each function, find the indicated function value.

a) $f(x) = -2x + 6;\ f(-1)$ b) $f(x) = 3x^2 + x - 5;\ f(0)$ c) $f(x) = 2|x| - 1;\ f(-3)$

Teaching Notes:

- Students find this section easy.
- Have students name different real-world relationships which represent functions.
- One way of viewing a function is to think of it as a box with an input and an output and a crank which takes as an input an element of the domain and produces as an output an element of the range.

Answers: *1a) –6, –6, (–3, –6), b) –5, –5, (–2, –5), c) –4, –4, (–1, –4), d) –3, –3, (0, –3), e) –2, –2, (1, –2); 2a)* $dm = \{1,2,4\}$, $rg = \{1,3,5\}$, *function, b)* $dm = \{-1,1,2\}$, $rg = \{3,-5\}$, *function, c)* $dm = \{3,7\}$, $rg = \{-1,4\}$, *not a function; 3a) function, b) not a function, c) function, d) not a function; 4a)* $f(-1) = 8$, *b)* $f(0) = -5$, *c)* $f(-3) = 5$

Evaluating Roots

Learning Objectives:

1. Find square roots.
2. Decide whether a given root is rational, irrational, or not a real number.
3. Find decimal approximations for irrational square roots.
4. Use the Pythagorean formula.
5. Find cubes, fourth, and other roots.

Examples:

1. Find the two square roots of each number.

   a) 9 b) 25 c) 36 d) 49 e) 100

2. Find the square root, if it exists. Do not use a calculator or a table of square roots.

   a) $\sqrt{4}$ b) $-\sqrt{4}$ c) $\sqrt{-4}$ d) $\sqrt{25}$

   e) $\sqrt{0.25}$ f) $-\sqrt{121}$ g) $-\sqrt{\frac{49}{36}}$ h) $\sqrt{\frac{81}{400}}$

3. Use a calculator or the square root table to approximate to the nearest thousandth.

   a) $\sqrt{5}$ b) $-\sqrt{7}$ c) $\sqrt{59}$ d) $-\sqrt{951}$

4. Draw a diagram and use the Pythagorean theorem to solve. Round to the nearest tenth.

   a) A 12-foot ladder is leaning against a house with the base of the ladder 4 feet from the house. How high up the house does the ladder reach?

   b) One end of a wire is attached to the top of a 25 foot pole, and the other end is anchored into the ground 20 feet from the base of the pole. Find the length of the wire.

5. Evaluate higher order roots.

   a) $\sqrt[3]{8}$ b) $-\sqrt[3]{8}$ c) $\sqrt[3]{-8}$

   d) $\sqrt[4]{16}$ e) $-\sqrt[4]{16}$ f) $\sqrt[4]{-16}$

Teaching Notes:

- Many students find the signs confusing with these problems.
- Although the square root of a negative number is included here, it is mentioned only briefly in the text and need not be a major focus point.
- Encourage students to memorize the squares of the numbers 1 through 12.

Answers: 1a) 3, –3, b) 5, –5, c) 6, –6, d) 7, –7, e) 10, –10; 2a) 2, b) –2, c) not a real number, d) 5, e) 0.5, f) –11, g) –7/6, h) 9/20; 3a) 2.236, b) –2.646, c) 7.681, d) –30.838; 4a) 11.3 feet, b) 32.0 feet; 5a) 2, b) –2, c) –2, d) 2, e) –2, f) not a real number

Multiplying, Dividing, and Simplifying Radicals

Learning Objectives:

1. Multiply square root radicals.
2. Simplify radicals using the product rule.
3. Simplify radicals using the quotient rule.
4. Simplify radicals involving variables.
5. Simplify other roots.

Examples:

1. Use the product rule and the quotient rule to find each product or quotient.

   a) $\sqrt{5}\cdot\sqrt{7}$ b) $\sqrt{6}\cdot\sqrt{7}$ c) $\sqrt{11}\cdot\sqrt{t}$ d) $\dfrac{\sqrt{252}}{\sqrt{7}}$

2. Simplify. Assume that all variables represent positive numbers.

   a) $\sqrt{3^2}$ b) $\sqrt{6^2}$ c) $\sqrt{2^4}$ d) $\sqrt{13^2}$

   e) $\sqrt{x^2}$ f) $\sqrt{x^4}$ g) $\sqrt{x^6}$ h) $\sqrt{x^{14}}$

   i) $\sqrt{25x^2}$ j) $\sqrt{49x^6y^2}$ k) $\sqrt{100x^{34}}$ l) $\sqrt{x^{44}y^{50}}$

3. Simplify. Assume that all variables represent positive numbers.

   a) $\sqrt{8}$ b) $-\sqrt{18}$ c) $\sqrt{72}$ d) $\sqrt{147}$

   e) $\sqrt{x^3}$ f) $\sqrt{x^5}$ g) $\sqrt{x^7}$ h) $\sqrt{x^{13}}$

   i) $\sqrt{80x^3}$ j) $\sqrt{25x^5y^2}$ k) $\sqrt{128x^6y^8}$ l) $\sqrt{63x^9y^3w^7}$

4. Simplifying other roots.

   a) $\sqrt[3]{24}$ b) $\sqrt[3]{54}$ c) $\sqrt[3]{40x^4y^7}$ d) $\sqrt[3]{-\dfrac{1}{512}}$

Teaching Notes:

- Many students have trouble with number 3.
- Encourage students to write non-perfect square numbers as the product of the highest possible perfect square and another number.
- Most students need a lot of practice finding square roots of variables with odd exponents.
- Refer students to the ***Product and Quotient Rule for Radicals*** chart in the textbook.

Answers: 1a) $\sqrt{35}$, b) $\sqrt{42}$, c) $\sqrt{11t}$ d) 6; 2a) 3, b) 6, c) 4, d) 13, e) x, f) x^2, g) x^3, h) x^7, i) $5x$, j) $7x^3y$, k) $10x^{17}$, l) $x^{22}y^{25}$; 3a) $2\sqrt{2}$, b) $-3\sqrt{2}$, c) $6\sqrt{2}$, d) $7\sqrt{3}$, e) $x\sqrt{x}$, f) $x^2\sqrt{x}$, g) $x^3\sqrt{x}$, h) $x^6\sqrt{x}$, i) $4x\sqrt{5x}$, j) $5x^2y\sqrt{x}$, k) $8x^3y^4\sqrt{2}$, l) $3x^4yw^3\sqrt{7xyw}$; 4a) $2\sqrt[3]{3}$, b) $3\sqrt[3]{2}$, c) $2xy^2\sqrt[3]{5xy}$, d) $-\frac{1}{8}$

Adding and Subtracting Radicals

Learning Objectives:

1. Add and subtract radicals.
2. Simplify radical sums and differences.
3. Simplify more complicated radical expressions.

Examples:

1. Combine, if possible. Do not use a calculator or a table of square roots.

   a) $3\sqrt{7}+2\sqrt{7}-\sqrt{7}$ b) $4\sqrt{3}-6\sqrt{2}+7\sqrt{3}+3\sqrt{2}$ c) $\sqrt{x}-3\sqrt{x}$

   d) $-2.3\sqrt{2a}+4.1\sqrt{2a}$ e) $3\sqrt{w}-2\sqrt{v}-\sqrt{w}$

2. Combine, if possible. Do not use a calculator or a table of square roots. Assume that all variables represent positive numbers.

   a) $\sqrt{12}+\sqrt{27}$ b) $5\sqrt{7}+3\sqrt{63}$ c) $\sqrt{50}-6\sqrt{98}+8\sqrt{72}$

   d) $\sqrt{36}+4\sqrt{72}-3\sqrt{12}$ e) $3\sqrt{48}-2\sqrt{8}+\sqrt{50}$ f) $9\sqrt{5y}-2\sqrt{20y}$

   g) $x\sqrt{x}+3\sqrt{x^3}$ h) $-4\sqrt{27y^3}+5y\sqrt{12y}$ i) $6x\sqrt{50x}-4\sqrt{18x^3}$

3. Solve the following applied problems. Do not use a calculator or a table of square roots.

   a) A rectangular plot of land has a length of $\left(24\sqrt{2}+6\sqrt{3}\right)$ yards and a width of $\left(2\sqrt{2}+6\sqrt{3}\right)$ yards. Find the perimeter of the plot of land.

   b) To suspend a prop from the ceiling on the set of the school play, Ernie needs 10 pieces of wire. Each of five of the pieces of wire needs a length of $\sqrt{54}$ meters, while each of the other five needs a length of $\sqrt{27}$ meters. Find the total length of wire required.

Teaching Notes:

- Most students find number 1 easy once they realize that adding/subtracting like radicals is analogous to adding/subtracting like terms.
- Many students have trouble at first with the examples in problem 2, where the square root has a coefficient other than 1 before simplification.

Answers: *1a)* $4\sqrt{7}$, *b)* $11\sqrt{3}-3\sqrt{2}$, *c)* $-2\sqrt{x}$, *d)* $1.8\sqrt{2a}$, *e)* $2\sqrt{w}-2\sqrt{v}$; *2a)* $5\sqrt{3}$, *b)* $14\sqrt{7}$, *c)* $11\sqrt{2}$, *d)* $6+24\sqrt{2}-6\sqrt{3}$, *e)* $12\sqrt{3}+\sqrt{2}$, *f)* $5\sqrt{5y}$, *g)* $4x\sqrt{x}$, *h)* $-2y\sqrt{3y}$, *i)* $18x\sqrt{2x}$; *3a)* $52\sqrt{2}+24\sqrt{3}$ *yards,* *b)* $15\sqrt{6}+15\sqrt{3}$ *meters*

Rationalizing the Denominator

Learning Objectives:

1. Rationalize denominators with square roots.
2. Write radicals in simplified form.
3. Rationalize denominators with cube roots.

Examples:

1. Simplify. Do not use a calculator or a table of square roots. Assume all variables represent positive numbers.

a) $\dfrac{\sqrt{20}}{\sqrt{5}}$ b) $\dfrac{\sqrt{6}}{\sqrt{54}}$ c) $\dfrac{\sqrt{12x^3}}{\sqrt{3x^2}}$ d) $\dfrac{\sqrt{27x^7}}{\sqrt{3x}}$

2. Rationalize the denominator. Simplify your answer. Assume all variables represent positive numbers.

a) $\dfrac{3}{\sqrt{11}}$ b) $\dfrac{\sqrt{6}}{\sqrt{x}}$ c) $\sqrt{\dfrac{13}{17}}$

d) $\dfrac{x\sqrt{x}}{\sqrt{5}}$ e) $\dfrac{3x}{\sqrt{x^5}}$ f) $\dfrac{\sqrt{32}}{\sqrt{2x^3}}$

3. Rationalize the denominator. Simplify your answer.

a) $\dfrac{4}{\sqrt[3]{2}}$ b) $\sqrt[3]{\dfrac{2x}{9y}}$ c) $\sqrt[3]{\dfrac{8x^3y^6}{9z}}$

Teaching Notes:

- Most students do not have trouble with number 1.
- For number 2, some students need to see several examples of how $\sqrt{a}\cdot\sqrt{a}=a$ before applying it to rationalizing a denominator.
- Sometimes putting radical expression in simplified form produces a more complicated looking expression.
- Remind students that when rationalizing a cube root, we must multiply by a radical that produces a perfect cube under the cube root sign in the denominator.

Answers: *1a) 2, b) 1/3, c)* $2\sqrt{x}$*, d)* $3x^3$*; 2a)* $\dfrac{3\sqrt{11}}{11}$*, b)* $\dfrac{\sqrt{6x}}{x}$*, c)* $\dfrac{\sqrt{221}}{17}$*, d)* $\dfrac{x\sqrt{5x}}{5}$*, e)* $\dfrac{3\sqrt{x}}{x^2}$*, f)* $\dfrac{4\sqrt{x}}{x^2}$*; 3a)* $2\sqrt[3]{4}$*,*
b) $\dfrac{\sqrt[3]{6xy^2}}{3y}$*, c)* $\dfrac{2xy^2\sqrt[3]{3z^2}}{3z}$

More Simplifying and Operations with Radicals

Learning Objectives:

1. Simplify products of radical expressions.
2. Use conjugates to rationalize denominators of radical expressions.
3. Write radical expressions with quotients in lowest terms.

Examples: (Assume all variables represent positive numbers.)

1. Multiply. Be sure to simplify any radicals in your answer. Do not use a calculator or a table of square roots.

a) $\sqrt{3}\sqrt{7}$ b) $\sqrt{3}\sqrt{33}$ c) $\sqrt{6}\sqrt{18}$

d) $\sqrt{6x}\sqrt{60x}$ e) $(2\sqrt{10x})(5\sqrt{5x})$ f) $(-6\sqrt{ab})(4\sqrt{b})$

2. Multiply. Be sure to simplify any radicals in your answer. Do not use a calculator or a table of square roots.

a) $\sqrt{5}(\sqrt{3}+\sqrt{7})$ b) $\sqrt{6}(4\sqrt{12}-3\sqrt{3})$ c) $-5\sqrt{b}(6\sqrt{a}+5\sqrt{b})$

d) $\sqrt{10}(2\sqrt{5}-2\sqrt{10}+6\sqrt{2})$ e) $2\sqrt{x}(\sqrt{y}+3\sqrt{xy}-5\sqrt{x})$

3. Multiply. Be sure to simplify any radicals in your answer. Do not use a calculator or a table of square roots.

a) $(2\sqrt{5}+\sqrt{3})(\sqrt{5}+\sqrt{3})$ b) $(2\sqrt{2}+5\sqrt{10})(2\sqrt{2}-\sqrt{10})$ c) $(3\sqrt{5}-2)^2$

d) $(4\sqrt{3}+6\sqrt{5})^2$ e) $(2a\sqrt{3}-3\sqrt{2})(2a\sqrt{3}+3\sqrt{2})$

4. Rationalize the denominator. Simplify your answer.

a) $\dfrac{3}{\sqrt{3}+1}$ b) $\dfrac{7}{\sqrt{13}-\sqrt{6}}$ c) $\dfrac{\sqrt{8}}{\sqrt{5}+\sqrt{8}}$

d) $\dfrac{3x}{2\sqrt{5}+2\sqrt{6}}$ e) $\dfrac{\sqrt{x}}{\sqrt{3}+\sqrt{2}}$ f) $\dfrac{5\sqrt{3}+2\sqrt{5}}{\sqrt{5}-\sqrt{3}}$

Teaching Notes:

- In examples 1e) and f), some students try to multiply using a FOIL approach instead of multiplying the coefficients and then multiplying the radicals.
- Many students distribute the exponent in examples 3c) and d) instead of multiplying the binomial by itself.

Answers: 1a) $\sqrt{21}$, b) $3\sqrt{11}$, c) $6\sqrt{3}$, d) $6x\sqrt{10}$, e) $50x\sqrt{2}$, f) $-24b\sqrt{a}$; 2a) $\sqrt{15}+\sqrt{35}$, b) $15\sqrt{2}$, c) $-30\sqrt{ab}-25b$, d) $10\sqrt{2}+12\sqrt{5}-20$, e) $2\sqrt{xy}+6x\sqrt{y}-10x$; 3a) $13+3\sqrt{15}$, b) $-42+16\sqrt{5}$, c) $49-12\sqrt{5}$, d) $228+48\sqrt{15}$, e) $12a^2-18$; 4a) $\dfrac{3\sqrt{3}-3}{2}$, b) $\sqrt{13}+\sqrt{6}$, c) $\dfrac{-2\sqrt{10}+8}{3}$, d) $\dfrac{-3x\sqrt{5}+3x\sqrt{6}}{2}$, e) $\sqrt{3x}-\sqrt{2x}$, f) $\dfrac{7\sqrt{15}+25}{2}$

Solving Equations with Radicals

Learning Objectives:

1. Solve radical equations having square root radicals.
2. Identify equations with no solutions.
3. Solve equations by squaring a binomial.
4. Solve problems using formulas that involve radicals.

Examples:

1. A right triangle has legs a and b and hypotenuse c. Find the length of the missing side. Leave any irrational answers in radical form.

   a) $a = 2, b = 3$. Find c. b) $a = \sqrt{3}$, $b = \sqrt{5}$. Find c.

   c) $a = 9, c = 15$. Find b. d) $b = \sqrt{3}$, $c = \sqrt{8}$. Find a.

2. Solve for the variable. Check your solutions.

   a) $\sqrt{x} = 7$ b) $\sqrt{y+3} = 6$ c) $\sqrt{10x-9} = 9$

   d) $\sqrt{8x-2} = \sqrt{x+4}$ e) $\sqrt{5x} - 3 = 7$ f) $\sqrt{5x-6} = x$

   g) $\sqrt{y+9} = y+3$ h) $\sqrt{12y-1} - 5y = y$ i) $\sqrt{x} + 3 = \sqrt{x+21}$

3. Three times the square root of 7 equals the square root of the sum of some number and 5. What is the number?

Teaching Notes:

- Some students are confused by finding a missing leg in 1c) and d).
- Encourage students to draw and label a diagram for the applied problems.
- Many students find numbers 2d) through i) difficult.
- Show students a simple example of an extraneous solution, such as:
  $x = 3 \rightarrow$ square both sides $\rightarrow x^2 = 9 \rightarrow x = \pm 3 \rightarrow x = -3$ is extraneous

Answers: *1a)* $\sqrt{13}$, *b)* $2\sqrt{2}$, *c) 12, d)* $\sqrt{5}$; *2a)* $\{49\}$, *b)* $\{33\}$, *c)* $\{9\}$, *d)* $\left\{\frac{6}{7}\right\}$, *e)* $\{20\}$, *f)* $\{2,3\}$, *g)* $\{0\}$ *(–5 extraneous), h)* $\left\{\frac{1}{6}\right\}$, *i)* $\{4\}$; *3) 58*